The laws describing the motion of the domain boundary, given by the relations (6) and (7), encompass a sufficiently broad collection of engineering heat- and mass-transfer problems.

It should be remarked that we can also apply the method in question to the system (1), wherein the latter is augmented by heat and matter sources; we can also apply it to a system containing, not two, but n transfer potentials.

NOTATION

T	is the temperature;
θ	is the moisture content;
Fo	is the Fourier number;
Ko, Ly, Pn	are the Kossovich, Lykov, and Posnov numbers, respectively;
$Ko^* = \varepsilon Ko$,	where ε is the factor of phase transition of a liquid into a vapor;
$\Gamma = 0, 1, 2$	for a plate, cylinder, and sphere, respectively.

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NUMERICAL ALGORITHM OF THE SOLUTION OF THE MULTIPHASE STEFAN PROBLEM

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A method is proposed for computing the temperature and position of the phase interface based on the passage to new variables and a new function. The transformation is invariant relative to the heat-conduction equation, and the boundaries in the new variables are fixed.

A whole series of papers on the Stefan problem exist, which are surveyed sufficiently completely in [1], and wherein a great deal of original material associated with the proof of the uniqueness and existence of the solution is also generalized. Numerical schemes for the solution are proposed in [2]. Significant attention is paid there to the mathematical aspect of the question, but no results are presented of practical tests or of computations. V. G. Melamed [3] also gave a numerical solution, realized in application to the case of freezing soils. Fundamental results of a cycle of the author's work are presented in [3]. An analogous problem in terms of physical content, but taking account of snow and the influence of the atmosphere, is considered in [4]. Let us note that the nature of the method of solution to be used is determined by the specifics of some definite problem to be solved, which is a particular case of the general Stefan problem. The present paper, which is oriented toward the hydrometeorology area from the viewpoint of practical applications, is organized in a similar plan.

We formulate the problem below. Let us examine the one-dimensional case. Between two fixed planes $z = 0$ and $z = H$ at a time $t = 0$ let there be n alternating layers of material in the liquid or solid aggregate state with the moving interfaces $z = h_m(t)$ $(m = 1, 2, \ldots, n-1)$, where phase transition occurs. Let one layer of another material whose outer boundary moves according to the known law $z = -l(t)$ also adjoin the surface $z = 0$. The initial temperature distribution is given in the whole domain $T^0(z)$. Let us consider the temperature a known

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function of the time on the bounding planes $z = -l(t)$ and $z = H$. For $z = 0$ it is natural to assume the condition to be satisfied to be the equality of the temperatures and the heat fluxes. Heat balance holds on the moving boundaries with the phase transition, and the temperature equals the phase-transition temperature T_0. Finally we will assume it equal to zero. In the general case this is always attained by the introduction of the difference between the sought temperature and T_0. The temperature field and the position of the phase interface are to be determined. We consider the coefficients λ and a to be time-dependent and discontinuous functions of z, where they remain constant within the limits of each layer. Written mathematically, this reduces to the following:

$$c_p\rho\frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\left(\lambda(z, t)\frac{\partial T}{\partial z}\right), \tag{1}$$

$$T|_{z=-l(t)} = \Phi_1(t), \quad (l(0) \neq 0), \tag{2}$$

$$T|_{z=-0} = T|_{z=+0}, \tag{3}$$

$$\lambda(z, t)\frac{\partial T}{\partial z}\Big|_{z=-0} = \lambda(z, t)\frac{\partial T}{\partial z}\Big|_{z=+0}, \tag{4}$$

$$T|_{z=h_m(t)-0} = T|_{z=h_m(t)+0} = 0, \quad (h_m(0) \neq 0), \tag{5}$$

$$(-1)^{m+1}\left[\lambda(z, t)\frac{\partial T}{\partial z}\Big|_{z=h_m(t)-0} - \lambda(z, t)\frac{\partial T}{\partial z}\Big|_{z=h_m(t)+0}\right] = A\frac{dh_m}{dt}, \tag{6}$$

$$T|_{z=H} = \Phi_2(t), \tag{7}$$

$$T|_{t=0} = T^\circ(z). \tag{8}$$

If the first layer is crystalline, then the even values of the exponent with -1 correspond to zones where the substance is in the solid aggregate state and the odd values, to zones where the substance is in the liquid state.

Now let us partition the whole time of interest to us into a number of steps $\Delta t = t^j - t^{j-1}$ ($j = 1, 2, \ldots$). Initially, we digress from the problem formulated above and examine some m-th layer in the j-th step, for which we consider the temperature of the moving boundaries to be known functions of time in the interest of generality.

Let us introduce the variables $\zeta = \zeta(z, t)$, $\tau = \tau(t)$ in place of z and t; let us also introduce the function

$$u(\zeta, \tau) = f(\zeta, \tau)\,\tilde{T}(\zeta, \tau) \tag{9}$$

in place of $T(z, t)$ (the superscript j will temporarily be omitted everywhere).

Henceforth, let us pose the problem of defining them in such a way that the boundaries would be fixed in the new variables and the equation for $u(\zeta, \tau)$ would have the form of the thermal-conductivity equation, i.e., we should have

$$\frac{\partial u}{\partial \tau} = b\,\frac{\partial^2 u}{\partial \zeta^2}. \tag{10}$$

Taking account of (9), this can be rewritten as

$$\frac{\partial f}{\partial \tau}\,\tilde{T} + f\,\frac{\partial \tilde{T}}{\partial \tau} = b\left(\frac{\partial^2 f}{\partial \zeta^2}\,\tilde{T} + 2\,\frac{\partial f}{\partial \zeta}\,\frac{\partial \tilde{T}}{\partial \zeta} + f\,\frac{\partial^2 \tilde{T}}{\partial \zeta^2}\right). \tag{11}$$

In turn, in the new variables in place of (1) we will have

$$\frac{\partial \tilde{T}}{\partial \tau} = a\left(\frac{\partial \zeta}{\partial z}\right)^2\frac{dt}{d\tau}\cdot\frac{\partial^2 \tilde{T}}{\partial \zeta^2} - \frac{dt}{d\tau}\cdot\frac{\partial \tilde{T}}{\partial \zeta}\left(\frac{\partial \zeta}{\partial t} - a\,\frac{\partial^2 \zeta}{\partial z^2}\right). \tag{12}$$

Substituting (12) into (11), we obtain

$$\left(\frac{\partial f}{\partial \tau} - b\,\frac{\partial^2 f}{\partial \zeta^2}\right)\tilde{T} - \left[2b\,\frac{\partial f}{\partial \zeta} + f\,\frac{dt}{d\tau}\left(\frac{\partial \zeta}{\partial t} - a\,\frac{\partial^2 \zeta}{\partial z^2}\right)\right]\frac{\partial \tilde{T}}{\partial \zeta} + $$

$$+ f\left[a\left(\frac{\partial \zeta}{\partial z}\right)^2\frac{dt}{d\tau} - b\right]\frac{\partial^2 \tilde{T}}{\partial \zeta^2} = 0. \tag{13}$$

It follows from (13) that the following relationships should be valid for compliance with the conditions posed above:

$$\frac{\partial f}{\partial \tau} = b \frac{\partial^2 f}{\partial \zeta^2}, \tag{14}$$

$$\frac{\partial \ln f}{\partial \zeta} = -\frac{1}{2b} \cdot \frac{dt}{d\tau} \left(\frac{\partial \zeta}{\partial t} - a \frac{\partial^2 \zeta}{\partial z^2} \right), \tag{15}$$

$$\left(\frac{\partial \zeta}{\partial z} \right)^2 \frac{dt}{d\tau} = \frac{b}{a}. \tag{16}$$

A deduction may be made from the form of (16) that ζ can be only a linear function of z; thus, we should write

$$\zeta = \alpha(t) z + \beta(t).$$

Requiring $\zeta = m-1$ if $z = h_{m-1}(t)$ and $\zeta = m$ if $z = h_m(t)$, we find

$$\zeta = \frac{z - h_m(t)}{\Delta h_m(t)} + m, \quad (m-1 \leqslant \zeta \leqslant m), \tag{17}$$

$$(\Delta h_m(t) = h_m(t) - h_{m-1}(t)).$$

Taking account of (17), from (16) we obtain

$$d\tau = \frac{a}{b} \cdot \frac{dt}{\Delta h_m^2(t)} \quad \text{or} \quad d\tau = \frac{a}{b} \cdot \frac{d(\Delta h_m)}{\Delta h_m^2} \cdot \frac{dt}{d(\Delta h_m)}.$$

This latter expression can be integrated if it is assumed that

$$\frac{d(\Delta h_m)}{dt} = \frac{\Delta h_m^j - \Delta h_m^{j-1}}{\Delta t}. \tag{18}$$

In this case we obtain

$$\tau = \frac{a\Delta t}{b(\Delta h_m^j - \Delta h_m^{j-1})} \left[\frac{\Delta h_m(t) - \Delta h_m^{j-1}}{\Delta h_m(t)\, \Delta h_m^{j-1}} \right]. \tag{19}$$

Since from (18) it results that

$$\Delta h_m(t) = \Delta h_m^{j-1} + (\Delta h_m^j - \Delta h_m^{j-1})(t - t^{j-1})/\Delta t, \tag{19'}$$

by using this latter expression we obtain a formula for τ of the form

$$\tau = \frac{a(t - t^{j-1})}{b} \cdot \frac{1}{\Delta h_m(t)\, \Delta h_m^{j-1}}. \tag{20}$$

Now let us define the constant b in (20) in such a way that for $t = t^j$ we would have $\tau = 1$. To do this, the equality

$$b = \frac{a\Delta t}{\Delta h_m^j \Delta h_m^{j-1}} \tag{21}$$

should be satisfied. Then, finally,

$$\tau = \frac{t - t^{j-1}}{\Delta t} \cdot \frac{\Delta h_m^j}{\Delta h_m(t)}, \quad (0 \leqslant \tau \leqslant 1). \tag{22}$$

To determine the form of the function $f(\zeta, \tau)$, let us use (15) together with (17), (21), and (22). After integrating (15), we arrive at the dependence

$$f(\zeta, \tau) = C(\tau) \exp \left\{ \frac{\psi(\tau)}{2} [(\zeta - m + \delta)^2 - (m - \delta)^2] \right\},$$

where

$$\psi(\tau) = \frac{\Delta h_m^j - \Delta h_m^{j-1}}{2b[\Delta h_m^j - (\Delta h_m^j - \Delta h_m^{j-1})\tau]}, \quad \delta = \frac{h_m^j - h_m^{j-1}}{\Delta h_m^j - \Delta h_m^{j-1}}, \tag{23}$$

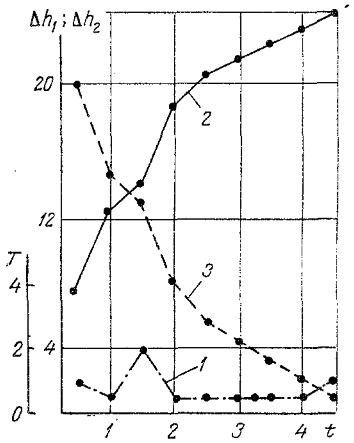

Fig. 1. Thawing soil: 1) soil surface temperature T (°C); 2) thawing-layer thickness Δh_1 (cm); 3) freezing-layer thickness Δh_2 (cm); t, days.

and C(τ) is an unknown function which we find by using (14) by substituting the last expression therein. Then

$$C(\tau)=\frac{C_1}{\sqrt{\Delta h_m^j-(\Delta h_m^j-\Delta h_m^{j-1})\tau}}\exp\left[\frac{\psi(\tau)(m-\delta)^2}{2}\right].$$

Hence, we obtain

$$f(\zeta,\tau)=\frac{C_1}{\sqrt{\Delta h_m^j(\Delta h_m^j-\Delta h_m^{j-1})\tau}}\exp\left\{\frac{(\Delta h_m^j-\Delta h_m^{j-1})(\zeta-m+\delta)^2}{4b[\Delta h_m^j-(\Delta h_m^j-\Delta h_m^{j-1})\tau]}\right\}. \tag{24}$$

The factor C_1 can be set equal to some constant value with the dimensionality of a length, which we select equal to Δh_m^{j-1}. We obtain the function u(ζ, τ) by substituting (24) into (9).

Different particular cases can be obtained from (17), (22), (23), and (24). Thus, we should put therein the following: for m = 0, $\Delta h_0(t)\equiv l(t)$, $l_0^j=l_0^{j-1}=0$, and δ = 0; for m = 1, $\Delta h_1(t)\equiv h_1(t)$ and δ = 1; for m = n, $\Delta h_n(t)=H-h_{n-1}(t)$ and δ = 0. The first and third of the cases considered correspond to plates for which the upper boundary moves and the lower is fixed and has the coordinates z = ζ = 0 or z = H, ζ = n; the second case corresponds to the presence of an upper fixed boundary, while the lower moves. Proceeding analogously, other particular cases can be obtained also, but for our purposes those presented above are sufficient because their combination will yield the problem posed.

Now let the quantity τ be the computational spacing. The velocity is kept constant within each time step, but changes during passage to the next step, i.e., in this case the curvilinear law of phase-interface motion is approximated by broken lines. Let us satisfy the boundary conditions for τ = 1. Then introducing the notation

$$u^j(\zeta,1)\equiv U^j(\zeta),\quad u^j(\zeta,0)=u^{j-1}(\zeta),$$
$$\tilde\lambda(\zeta,1)\equiv\lambda^j(\zeta),\quad \tilde a(\zeta,\tau)\equiv a^j(\zeta),$$

we will have

$$U^j-u^{j-1}=b^j(\zeta)\frac{\partial^2U^j}{\partial\zeta^2}, \tag{25}$$

$$U^j|_{\zeta=-1}=\exp\left[\frac{l^j-l^{j-1}}{4b^j(-1)\,l^{j-1}}\right]\tilde\Phi_1^j(1), \tag{26}$$

$$U^j|_{\zeta=-0}=U^j|_{\zeta=+0}, \tag{27}$$

$$\frac{\lambda^j(\zeta)}{l^j}\cdot\left.\frac{\partial U^j}{\partial\zeta}\right|_{\zeta=-0}=\frac{\lambda^j(\zeta)}{h^j}\cdot\left.\frac{\partial U^j}{\partial\zeta}\right|_{\zeta=+0}, \tag{28}$$

$$U^j|_{\zeta=m-0}=U^j|_{\zeta=m+0}=0, \tag{29}$$

$$\frac{\lambda^j(\zeta)}{\Delta h_m^j}\exp\left[-\frac{(\Delta h_m^j-\Delta h_m^{j-1})\,\delta^2}{4b^j(\zeta)\,\Delta h_m^{j-1}}\right]\frac{\partial U^j}{\partial\zeta}\bigg|_{\zeta=m-0}-\frac{\lambda^j(\zeta)}{\Delta h_{m+1}^j}\times$$

$$\times\exp\left[-\frac{(\Delta h_{m+1}^j-\Delta h_{m+1}^{j-1})\,(\delta+1)^2}{4b^j(\zeta)\,\Delta h_{m+1}^{j-1}}\right]\frac{\partial U^j}{\partial\zeta}\bigg|_{\zeta=m+0}=A\,\frac{\Delta h^{j+1}}{\Delta t}. \tag{30}$$

$$U^j|_{\zeta=n}=\tilde{\Phi}_2^j(1). \tag{31}$$

The conversion is carried out for $j > 1$. Thus, if U^{j-1} is the solution for the $j-1$-th stem for $\tau = 1$, then we will have in the j-th step for $\tau = 0$

$$u^{j-1}(\zeta)=\sqrt{\frac{\Delta h_m^{j-1}}{\Delta h_m^j}}\exp\left\{\frac{(\zeta-m+\delta)^2}{4}\left[\frac{\Delta h_m^j-\Delta h_m^{j-1}}{b^j(\zeta)\,\Delta h_m^{j-1}}-\frac{\Delta h_m^{j-1}-\Delta h_m^{j-2}}{b^{j-1}(\zeta)\,\Delta h_m^{j-2}}\right]\right\}U^{j-1}(\zeta). \tag{32}$$

The expression (32) plays the part of initial conditions in the j-th step. No conversion is carried out in the first step, but the quantity $u^0(\zeta)$ can be obtained from (9) and (24) for $\tau = 0$, so that

$$u^0(\zeta)=\sqrt{\frac{\Delta h_m^0}{\Delta h_m^1}}\exp\left[\frac{(\Delta h_m^1-\Delta h_m^0)(\zeta-m+\delta)^2}{4b^0(\zeta)\,\Delta h_m^1}\right]\tilde{T}^0(\zeta). \tag{32'}$$

Therefore, it can be stated that the problem with moving boundaries is in no way different from ordinary problems on heat propagation in composite plates with fixed planes of separation in its formal description [without (30)] because of the transformations mentioned. Hence, any of the classical numerical schemes is applicable for its solution. The single singularity is that a conversion of the initial conditions in conformity with (32) must be performed in each step. But this produces no difficulties and does not complicate the problem in practice. Condition (30) is used to seek h^{j+1} after which the quantity $U^j(\zeta)$ is found as a result of solving (25) in combination with the conditions (26)-(29), (31)-(32). If the first and second members in the left side of (30) are denoted, respectively, by k_1^j and k_2^j, then we obtain at once

$$h^{j+1}=h^j+\frac{(k_1^j-k_2^j)\,\Delta t}{A}. \tag{33}$$

The quantity h^{j+1} found from (33) is used to calculate $U^{j+1}(\zeta)$, after which the process is repeated so that we obtain h^{j+2}, etc. For $j = 1$, we must resort to iteration to determine h^1. Initially setting $h_1^1 = h^0$, we find $U_1^1(\zeta)$ and then h_2^1 from (33). Then we again repeat the whole procedure using h_2^1, so that we finally obtain h_3^1. The process is continued until the inequality $|h_{n-1}^1-h_n^1| < \varepsilon$ becomes valid for two values found successively, where ε is a previously assigned small number. The quantity h_n^1 is taken as the true h^1.

Let us note that we could operate analogously in the next steps, which would correspond to an implicit scheme for computing h^1, but as numerical experiments have shown, the implicit scheme elucidated above yields completely satisfactory accuracy, while it is simultaneously less tedious. Hence, it was taken as basic.

A number of numerical tests were carried out to verify the method proposed. Consider the freezing and thawing of snow-covered soil. The results of a computation and of observations in nature are in good agreement. Data obtained in computing the thawing, for which materials of an expedition of the Voeikov Main Geophysical Observatory to the Tsimlyansk reservoir (in the region of the collective farm "Gigant") were used, are presented in Fig. 1. For a positive air temperature, the temperature on the snow surface was zero and a three-layered medium was considered: snow ($m = 0$), frozen soil ($m = 1$), and thawed soil ($m = 2$). At the $H = 2$m level, $\Phi_2(t) = 8°C$, which corresponds to the mean multiannual soil temperature. After the snow vanishes, a thawing layer occurs in the soil from the top so that there are two thawing layers (from above and from below) and a freezing layer in between. As an analysis shows, the total time for disappearance of the frozen layer, starting from the time the snow falls, is ~4.5 days, which corresponds to actual measurements. Curves 2 and 3 in Fig. 1 illustrate the behavior of the moving frozen-layer boundaries.

The quantity $A = L\gamma[\tilde{w}^0(\zeta)|_{\zeta=1}-w_0]$ was used in the computation. Values of the thermophysical coefficients were taken from handbooks.

All the above refers to a problem with boundary conditions of the first kind on the outer surfaces, but the algorithm developed can, in principle, be used even in the presence of boundary conditions of the second and third kinds.

In conclusion, let us note that the method proposed can be modified. Thus, if the outer given temperature varies sufficiently smoothly during a long time, then there is every foundation to consider that the relationship (19) remains valid in this interval. The problem can then be solved by partitioning τ into a number of finer sections which will be the computation steps, and the value of the temperature, obtained in the previous interval, just for the value $\tau = 1$, will be used when going over to the next value of h_m^j. In the case mentioned, such an approach is more efficient.

NOTATION

t	is the time;
z	is the coordinate;
T(z, t)	is the temperature;
$\lambda(z, t)$, $a(z, t)$	are the coefficients of thermal conductivity and thermal diffusivity, respectively;
$h_m(t)$	are the coordinates of the phase-interface position;
H	is the lower boundary coordinate;
L	is the heat of the phase transition;
γ	is the volume weight of the soil;
$w^0(z)$	is the given moisture distribution in the soil;
w_0	is the experimentally determined quality of moisture which does not freeze at 0°C.

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SEMI-ANALYTICAL ALGORITHM FOR THE APPROXIMATE SOLUTION OF A NONSTATIONARY INVERSE PROBLEM OF DIFFUSION ON THE BASIS OF A DIRECT METHOD OF SOLUTION, LINEAR PROGRAMMING, AND REGULARIZATION METHODS

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Some analytical solutions of the direct problem of diffusion are presented for infinite bodies. The direct solutions constructed are used in algorithms for the approximate solution of the nonstationary inverse diffusion problem.

Results directly concerning the process of diffusion scattering of a substance are elucidated below. However, because of the analogy between the thermal conduction and diffusion processes, the results obtained are automatically carried over to the contiguous thermal-conductivity problem.

Let $0\xi\eta\zeta$ and 0xyz be the combined Cartesian reference systems with the ζ and z axes directed downward.

Let us consider the free diffusion process in a half-space (in the absence of sources and sinks):

$$V = \{(\xi, \eta, \zeta) : |\xi| < \infty, |\eta| < \infty, \zeta \geqslant 0\}. \tag{1}$$

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